Realizing Public Announcements by Justifications

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Abstract

Modal public announcement logics study how beliefs change after public announcements. However, these logics cannot express the reason for a new belief. Justification logics fill this gap since they can formally represent evidence and justifications for an agent's belief. We present $\mathsf{OPAL}(\mathsf{K})$ and $\mathsf{JPAL}(\mathsf{K})$, two alternative justification counterparts of Gerbrandy–Groeneveld's public announcement logic $\mathsf{PAL}(\mathsf{K})$. We show that $\mathsf{PAL}(\mathsf{K})$ is the forgetful projection of $\mathsf{OPAL}(\mathsf{K})$, respectively of $\mathsf{JPAL}(\mathsf{K})$, and we establish that $\mathsf{JPAL}(\mathsf{K})$ partially realizes $\mathsf{PAL}(\mathsf{K})$. The question whether a similar result also holds for $\mathsf{OPAL}(\mathsf{K})$ is still open.

1 Introduction

Justification logics are epistemic logics that feature explicit reasons for an agent's knowledge and belief. Instead of an implicit statement $\Box A$ that stands for *the agent knows* A, justification logics include explicit statements of the form $t : A$ that mean t *justifies the agent's knowledge of* A. In these statements, the evidence term t may represent a formal mathematical proof of A or an informal reason for A.

Originally, Artemov developed justification logic to provide a constructive semantics for intuitionistic logic. Later these logics were introduced into formal epistemology where they provide a novel approach to certain epistemic puzzles and to several problems of multi-agent systems [1, 2, 3, 4, 7].

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Dynamic epistemic logic [20] studies the relationship between communication, knowledge, and belief. It is based on the language of modal logic enriched with statements to express various forms of communication. A basic form of communication is provided by *public announcements* where a statement A is publicly communicated to all the agents. The logic of public announcements [16, 13] uses a statement $[A]B$ to express that B holds after the public announcement of A.

In this paper, we are interested in belief rather than in knowledge and, hence, rely on Gerbrandy–Groeneveld's axiomatization of public announcements [13]. One of its postulates is

$$\Box(A \to [A]B) \leftrightarrow [A]\Box B \ , \tag{1}$$

which says, from left to right, that an agent who believes that B must be the case whenever a true fact A is announced will believe B after an actual announcement of A.

To illustrate how this principle works, we briefly recall the following example from [8]. Elite-level frequent flyers can usually check in for their flight at the business counter by presenting their elite membership card, which can also be attached to their luggage to make public their elite status. This check-in rule is known to airline employees. In this situation, it follows by the implication (1) that when Ann presents her elite membership card to Bob at the business counter, he knows that he should check her in.

Modal public announcement logic tells us *how* beliefs change after public announcements but not *why*. It is the aim of this paper to formalize possible answers to this *why* using the approach of justification logic. If we convert the left to right implication from (1) to a statement with explicit justifications, we obtain something like

$$s : (A \to [A]B) \to [A]t : B \ , \tag{2}$$

where s represents the airline's regulations regarding business-counter check-in procedures and t is the reason why Bob starts checking Ann in.

The question is how the terms s and t, which represent justifications, relate to each other; in particular, how to arrive at t given s. We use the above example to discuss different answers to this question. There are the following possibilities.

1. $t = s$. The regulations themselves tell Bob to check Ann in. This option is implemented in the logic $\mathsf{JPAL}(\mathsf{K})$, which we developed jointly with Bryan Renne and Joshua Sack [6].

2. $t = \Uparrow s$. The operator $\Uparrow$ represents the inference Bob has to make from the regulations after the elite card is shown. This approach is taken by the logic $\mathsf{OPAL(K)}$, which we introduced in [8].

3. $t = \Uparrow_A s$. The inference process explicitly mentions both the regulations, s, and the demonstration of Ann's elite card, A. We do not consider this variant since it would make schematic reasoning impossible. Indeed (1) is an axiom scheme that does not depend on the announcement. Therefore, the operation that represents the update on the level of terms should not depend on the announcement either.

As already argued in [8], the simplicity of the first option, axiomatized by $\mathsf{JPAL(K)}$, may not always be sufficient. Imagine that Ann has been upgraded to business class (say, as a reward for postponing her original flight, which had been overbooked). So, according to the same regulations, she can check in with Bob based on her ticket alone without announcing her elite status, which in our notation is represented by $s : B$. But Ann may choose to announce her elite status anyways, or $[A]s : B$ in our notation. In $\mathsf{JPAL(K)}$, where $t = s$, after the elite status is announced, t encodes two different reasons for Bob to check Ann in: as a business-class customer and as an elite flyer. By contrast, in $\mathsf{OPAL(K)}$, these two reasons are represented by two different terms, s and $\Uparrow s$, of which the latter depends on Ann's elite status while the former is due to the ticket alone. And Bob would want to distinguish between the two reasons because of the difference in baggage allowances: an elite frequent flyer is often allowed to check more luggage for free than an owner of a business-class ticket who has been upgraded from economy.

In addition, in this and similar cases, the approach of $\mathsf{JPAL(K)}$ implies that the meaning of the regulations changes after public announcements: if Ann has an economy ticket, the regulations do not allow her a business-counter check-in until she shows her elite card, and then they do. This is a little counterintuitive since the regulations are a legal document whose meaning should not be changed by each public announcement. The use of reason $\Uparrow s$ enables us to separate the permanent status of the regulations from their momentary applications influenced by public announcements.

Let us now look at the other direction of (1)—from right to left—and see how the first two options manifest themselves there. The implication states that an agent who will believe B after an announcement of A must believe that, if A is true and announced, B holds after the announcement. For instance, if Charlie, while standing in a long line at the economy check-in counter, sees Ann showing her elite card and being served by Bob at the business counter, $[A]\Box B$, then Charlie has empirical evidence e that

Ann is served at the business counter, $[A]e : B$. It would be natural for Charlie to believe that having an elite status and showing it gets one to the business counter, $\Box(A \to [A]B)$. But it seems even clearer in this case that Charlie's empirical observation e cannot explain the causality of the implication $A \to [A]B$. If before Ann showed up, Charlie had read the sign that invited elite members to the business counter, then Charlie's memory of this sign, refreshed by Ann's actions, could serve as such an explanation. Thus, instead of using e, as in $\mathsf{JPAL}(\mathsf{K})$, in this example it also seems better to use $\Downarrow e$, where $\Downarrow$ is yet another new operation of our logic $\mathsf{OPAL}(\mathsf{K})$.

So far, not much work has been done to provide explicit justifications for dynamic epistemic logic. Besides the already mentioned [6, 8], there is Renne's earlier research on introducing new evidence [17] and eliminating unreliable evidence [19] in the framework of justification logic. He also presents expressivity results for certain justification logics with announcements [18].

However, the modal counterparts of Renne's systems do not correspond to any traditional public announcement logic whereas both $\mathsf{JPAL}(\mathsf{K})$ and $\mathsf{OPAL}(\mathsf{K})$ are intended as justification logics with public announcement operators whose belief dynamics closely corresponds to the modal belief dynamics of Gerbrandy–Groeneveld's modal public announcement logic $\mathsf{PAL}(\mathsf{K})$ [13].

In the next section, we recall the axiomatization and basic properties of $\mathsf{PAL}(\mathsf{K})$. In particular, we present the reduction of a $\mathsf{PAL}(\mathsf{K})$ formula A to a provably equivalent formula $\mathsf{red}(A)$ that does not contain public announcement operators. This reduction facilitates a simple completeness proof for $\mathsf{PAL}(\mathsf{K})$ by reducing it to completeness of the basic modal logic K.

As mentioned before, $\mathsf{JPAL}(\mathsf{K})$ and $\mathsf{OPAL}(\mathsf{K})$ (both with additional positive introspection axioms) were introduced in [6] and [8] respectively where we also established soundness and completeness for these two logics. We recall the definitions of $\mathsf{OPAL}(\mathsf{K})$ and $\mathsf{JPAL}(\mathsf{K})$ and their semantics in Sections 3 and 4. We reprove soundness and completeness for $\mathsf{OPAL}(\mathsf{K})$ and $\mathsf{JPAL}(\mathsf{K})$ in Section 5. Since the replacement property does not hold in justification logics, we cannot establish completeness of either logic by reducing it to completeness of the basic justification logic J. Instead we perform a canonic model construction for each of the two logics.

Section 6 is the main part of the paper. It deals with the formal relationship of $\mathsf{PAL}(\mathsf{K})$ and $\mathsf{JPAL}(\mathsf{K})$, respectively $\mathsf{OPAL}(\mathsf{K})$. The relationship is described by means of the notion of *forgetful projection*: given a justification logic formula A, its forgetful projection A° is a modal formula that is given by replacing each evidence term in A with $\Box$. We get the following theorem.

Theorem (Forgetful Projection)**.** For all justification logic formulas A,

$$\begin{aligned} \mathsf{JPAL(K)} \vdash A \quad &\Longrightarrow \quad \mathsf{PAL(K)} \vdash A^\circ \ , \\ \mathsf{OPAL(K)} \vdash A \quad &\Longrightarrow \quad \mathsf{PAL(K)} \vdash A^\circ \ . \end{aligned}$$

More interesting and much more difficult is the converse direction. To formulate it in a precise way, we need the notion of *realization*: given a modal formula A, a realization of A is a justification logic formula $r(A)$ such that $(r(A))^\circ = A$. We obtain the following theorem.

Theorem (Realization)**.** For all modal formulas A that do not contain modalities within announcements,

$$\mathsf{PAL(K)} \vdash A \quad \Longrightarrow \quad \mathsf{JPAL(K)} \vdash r(A) \text{ for some realization } r(A) \text{ of } A.$$

We establish the Realization Theorem in the following way. First, we reduce the $\mathsf{PAL(K)}$ formula A to a provably equivalent formula $\mathsf{red}(A)$ that has no announcement operators, i.e., $\mathsf{red}(A)$ is a traditional modal logic formula. Then we use realization for modal logic (without public announcements) to obtain a justification logic formula $r(\mathsf{red}(A))$ that realizes $\mathsf{red}(A)$. Finally, we 'invert' the reduction from A to $\mathsf{red}(A)$ on the justification logic side to obtain a formula $r(A)$ that realizes A.

$$\begin{tikzcd} A \arrow[d, "\text{Reduction}"'] & r(A) \arrow[l, dashed, "\text{Forgetful projection}"'] \\ \mathsf{red}(A) \arrow[r, "\mathsf{K}\text{ realization}"'] & r(\mathsf{red}(A)) \arrow[u, "\text{Replacement to `invert' }\mathsf{red}"'] \end{tikzcd}$$

This *realization by reduction* approach is a novel technique in the realm of justification logic. The closest analog of this method can be found in [11], where $\mathsf{S5}$ is realized by reducing it to $\mathsf{K45}$. However, there the reversal of the reduction is trivial, while in our setting it requires an involved extension of Fitting's replacement theorem [10]. First, we need replacement also for formulas with public announcements and, second, we need replacement also in negative positions (the original proof in [10] only deals with replacement in positive positions). While we only show this extended replacement theorem for $\mathsf{JPAL(K)}$, there seems to be little or no extra work required to prove the same extended replacement theorem for $\mathsf{OPAL(K)}$. The problem lies in the application of this replacement theorem to reverse the modal reduction on the justification side for $\mathsf{OPAL(K)}$. The exact nature of the problem is too technical to be explained in the introduction and is pointed out in the proof of

Theorem 43, Footnote 4. We only mention here that the problem concerns reversing in $\mathsf{OPAL}(\mathsf{K})$ the modal update reduction in a negative position. Thus, we obtain realization only for $\mathsf{JPAL}(\mathsf{K})$. It is open whether a similar result can be shown for $\mathsf{OPAL}(\mathsf{K})$.

The replacement theorem requires that there exist realization functions that satisfy the technical condition of *non-self-referentiality on variables.* By a result of Kuznets [14], we know that such realization functions exist for the modal logic K. Therefore, we formulate $\mathsf{OPAL}(\mathsf{K})$ and $\mathsf{JPAL}(\mathsf{K})$ as justification counterparts of public announcement logic over K. The original versions of these two logics in [6, 8] included axioms for positive introspection, and we claimed in [8] that realization holds for JPAL with positive introspection. However, it is not clear whether there always exist realization functions for positive introspection (i.e., $\mathsf{K4}$) that are non-self-referential on variables. Thus, it is still open whether a realization theorem holds for public announcements in the presence of positive introspection.

The results presented in this paper are based on the conference papers [6, 8] that did not include full proofs.

2 Modal Public Announcement Logic

In this section, we recall some of the basic definitions and facts concerning the Gerbrandy–Groeneveld modal logic of public introspective announcements [12, 13, 20], i.e, public announcements that need not be truthful but are trusted by all the agents.

Definition 1 ($\mathsf{PAL}(\mathsf{K})$ Language)**.** We fix a countable set Prop of *atomic propositions.* The *language of* $\mathsf{PAL}(\mathsf{K})$ consists of the *formulas* $A \in \mathsf{Fml}_{\Box,[\cdot]}$ formed by the grammar

$$A ::= p \mid \neg A \mid (A \to A) \mid \Box A \mid [A]A \qquad p \in \mathsf{Prop}$$

The language $\mathsf{Fml}_{\Box}$ of modal formulas without announcements is obtained from the same grammar without the $[A]A$ constructor.

The Gerbrandy–Groeneveld theory $\mathsf{PAL}(\mathsf{K})$ of Public Announcement Logic uses the language $\mathsf{Fml}_{\Box,[\cdot]}$ to reason about belief change and public announcements.

Definition 2 ($\mathsf{PAL}(\mathsf{K})$ Deductive System)**.** The *axioms of* $\mathsf{PAL}(\mathsf{K})$ consist of all $\mathsf{Fml}_{\Box,[\cdot]}$-instances of the following schemes:

1. Axiom schemes for the modal logic K

2. $[A]p \leftrightarrow p$ (independence)
3. $[A](B \to C) \leftrightarrow ([A]B \to [A]C)$ (normality)
4. $[A]\neg B \leftrightarrow \neg[A]B$ (functionality)
5. $[A]\Box B \leftrightarrow \Box(A \to [A]B)$ (update)
6. $[A][B]C \leftrightarrow [A \land [A]B]C$ (iteration)

The *deductive system* $\mathsf{PAL(K)}$ is a Hilbert system that consists of the above axioms of $\mathsf{PAL(K)}$ and the following rules of *modus ponens* (MP) and *necessitation* (N):

$$\frac{A \quad A \to B}{B}\,\text{(MP)} \quad , \qquad \frac{A}{\Box A}\,\text{(N)} \quad .$$

We write $\mathsf{PAL(K)} \vdash A$ to state that $A \in \mathsf{Fml}_{\Box,[\cdot]}$ is a theorem of $\mathsf{PAL(K)}$.

Lemma 3 (Admissible Announcement Necessitation, [20])**.** *Announcement necessitation is admissible in* $\mathsf{PAL(K)}$*: that is, for all formulas* $A, B \in \mathsf{Fml}_{\Box,[\cdot]}$*, we have*

$$\mathsf{PAL(K)} \vdash A \textit{ implies } \mathsf{PAL(K)} \vdash [B]A \enspace .$$

$\mathsf{PAL(K)}$, like many traditional modal public announcement logics, features the so-called *reduction property*: $\mathsf{Fml}_{\Box,[\cdot]}$-formulas can be reduced to provably equivalent $\mathsf{Fml}_{\Box}$-formulas [12, 13, 20]. That means one can express what the situation is after an announcement by saying what the situation was before the announcement. The following lemma formally describes this reduction procedure (for a proof, see, for instance, [20]). The proof method was first introduced by Plaza in [16].

Definition 4 (Announcement Redexes and their Reducts)**.** The following are five pairs of redexes and their reducts:

Redex	Its reduct
$[A]p$	p
$[A]\neg B$	$\neg[A]B$
$[A](B \to C)$	$[A]B \to [A]C$
$[A]\Box B$	$\Box(A \to [A]B)$
$[A][B]C$	$[A \land [A]B]C$

(3)

Definition 5 (Reduction)**.** The one-step reduction function $\mathsf{red}_1 : \mathsf{Fml}_{\Box,[\cdot]} \to \mathsf{Fml}_{\Box,[\cdot]}$ is defined as follows: If no subformula of $A \in \mathsf{Fml}_{\Box,[\cdot]}$ is a redex, then $\mathsf{red}_1(A) := A$. Otherwise, let R be the outermost leftmost subformula occurrence of A that is a redex, i.e.,

1) if R is a proper subformula occurrence of R', which is a subformula occurrence of A, R' is not a redex;
2) if R is a subformula occurrence of C and $B \to C$ is a subformula occurrence of A, no redex can occur in B.
In this case, $\mathsf{red}_1(A)$ is defined as the result of replacing the formula occurrence R in A with its reduct. Note that R is outside of announcements in A. Indeed, if R occured in B with $[B]C$ being a subformula occurrence of A, then $[B]C$ would itself be a redex, which is prohibited by item 1) above.

It is easy to show, using a formula rank similar to the one from Definition 28, that for any formula $A \in \mathsf{Fml}_{\Box,[\cdot]}$, there exists $N > 0$ such that $\mathsf{red}_1^{N+1}(A) = \mathsf{red}_1^{N}(A)$, which must then be an $\mathsf{Fml}_{\Box}$-formula. In this case, $\mathsf{red}_1^{n}(A) = \mathsf{red}_1^{N}(A)$ for any $n \geq N$ and we define $\mathsf{red}(A) := \mathsf{red}_1^{N}(A)$.

Lemma 6 (Provable Equivalence of Reductions)**.** *For all* $A \in \mathsf{Fml}_{\Box,[\cdot]}$, *we have* $\mathsf{PAL(K)} \vdash A \leftrightarrow \mathsf{red}_1(A)$, *and, consequently,* $\mathsf{PAL(K)} \vdash A \leftrightarrow \mathsf{red}(A)$.

Remark 7. The above lemma facilitates a completeness proof for $\mathsf{PAL(K)}$ by reducing it to completeness of K. The completeness is proved with respect to the class of all Kripke models. To evaluate validity of formulas with announcements, the standard Kripke semantics is extended with a model update operation for introspective announcements. To save space, instead of formulating this semantics, we refer the reader to [20, Section 4.9] and give only a sketch of the completeness proof. Suppose that $A \in \mathsf{Fml}_{\Box,[\cdot]}$ is valid. Then $\mathsf{red}(A)$ is also valid by Lemma 6 and by soundness of $\mathsf{PAL(K)}$, which is easy to show directly. Since $\mathsf{red}(A)$ is a formula of $\mathsf{Fml}_{\Box}$, completeness of K yields $\mathsf{K} \vdash \mathsf{red}(A)$ and, hence, $\mathsf{PAL(K)} \vdash \mathsf{red}(A)$ because $\mathsf{PAL(K)}$ extends K. Applying Lemma 6 again, we conclude that $\mathsf{PAL(K)} \vdash A$. As a corollary of the soundness of $\mathsf{PAL(K)}$, since the semantics for $\mathsf{PAL(K)}$ extends the standard Kripke semantics, $\mathsf{PAL(K)}$ is a conservative extension of K.

3 Justification Logic

Our language extends the language typically used in justification logic by adding public announcement formulas $[A]B$ and two unary operations on terms, $\Uparrow$ and $\Downarrow$, to express the update dynamics of evidence.

Definition 8 (Language)**.** In addition to the set of propositions Prop, we fix countable sets Cons of *constants* and Vars of *variables*. Our language consists of the *terms* $t \in \mathsf{Tm}$ and the *formulas* $A \in \mathsf{Fml}_{\mathsf{J}}$ formed by the grammar

$$
\begin{array}{rcll}
t & ::= & x \mid c \mid (t \cdot t) \mid (t + t) \mid {\Uparrow} t \mid {\Downarrow} t & \qquad x \in \mathsf{Vars},\ c \in \mathsf{Cons} \\
A & ::= & p \mid \neg A \mid (A \to A) \mid t : A \mid [A]A & \qquad p \in \mathsf{Prop}
\end{array}
$$

A term is a *ground term* if it does not contain variables. The language introduced in [6, 8] for justification logics with public announcements includes additionally an operation ! on terms that is used for positive introspection. Since the logics $\mathsf{OPAL}(\mathsf{K})$ and $\mathsf{JPAL}(\mathsf{K})$, to be introduced below, do not have an introspection axiom, we can dispense with the ! operation.

Remark 9. To state axioms of our systems $\mathsf{JPAL}(\mathsf{K})$ and $\mathsf{OPAL}(\mathsf{K})$, we use arbitrary finite sequences of announcements, which is not done in modal public announcement logics. This use of sequences may seem puzzling, especially given that the iteration axiom of $\mathsf{PAL}(\mathsf{K})$, which normally allows to replace any such finite sequence with a single announcement, is transferred to $\mathsf{JPAL}(\mathsf{K})$ and $\mathsf{OPAL}(\mathsf{K})$ as is. But recall that the replacement property does not hold for justification logics, as already mentioned earlier. Replacing single announcements with sequences of announcements in axioms is the minimally invasive solution we have found to ensure the admissibility of the announcement necessitation rule, which is clearly valid semantically. For instance, if a theorem B is obtained by modus ponens from $C \to B$ and C, it follows by announcement necessitation that $[A_1] \dots [A_n]B$ should be derivable. For $n = 1$, the normality axiom of $\mathsf{PAL}(\mathsf{K})$ takes care of the transition from $[A_1](C \to B)$ and $[A_1]C$ to $[A_1]B$. In order to make such a transition possible for an arbitrary $n > 0$ we generalize the normality axiom to allow an arbitrary finite sequence of announcements $[A_1] \dots [A_n]$.

Notation 10 (Sequences of Announcements)**.** σ and τ (with and without subscripts) will denote finite sequences of formulas. ε denotes the empty sequence. Given such a sequence $\sigma = (A_1, \dots, A_n)$ and a formula B, the formula $[\sigma]B$ is defined as follows:

$$[\sigma]B := [A_1] \dots [A_n]B \text{ if } n > 0 \qquad \text{and} \qquad [\varepsilon]B := B.$$

Further, we define $\sigma, B := (A_1, \dots, A_n, B)$ and $B, \sigma := (B, A_1, \dots, A_n)$. For a sequence $\tau = (C_1, \dots, C_m)$, we define $\tau, \sigma := (C_1, \dots, C_m, A_1, \dots, A_n)$. We will also need the length $|\sigma|$ of a sequence σ, which is given by $|\varepsilon| := 0$ and $|(A_1, \dots, A_n)| := n$.

Definition 11 ($\mathsf{OPAL}(\mathsf{K})$ Deductive System)**.** The *axioms of* $\mathsf{OPAL}(\mathsf{K})$ consist of all Fml_J-instances of the following schemes:

1. $[\sigma]A$, where A is a classical propositional tautology
2. $[\sigma](t : (A \to B) \to (s : A \to t \cdot s : B))$ (application)
3. $[\sigma](t : A \to t + s : A)$, $\quad [\sigma](s : A \to t + s : A)$ (sum)
4. $[\sigma]p \leftrightarrow p$ (independence)

5. $[\sigma](B \to C) \leftrightarrow ([\sigma]B \to [\sigma]C)$ (normality)
6. $[\sigma]\neg B \leftrightarrow \neg[\sigma]B$ (functionality)
7. $[\sigma]t : (A \to [A]B) \to [\sigma][A] \Uparrow t : B$ (update $\Uparrow$)
8. $[\sigma][A]t : B \to [\sigma] \Downarrow t : (A \to [A]B)$ (update $\Downarrow$)
9. $[\sigma][A][B]C \leftrightarrow [\sigma][A \wedge [A]B]C$ (iteration)

The *deductive system* $\mathsf{OPAL(K)}$ is a Hilbert system that consists of the above axioms of $\mathsf{OPAL(K)}$ and the following rules of *modus ponens* (MP) and *axiom necessitation* (AN):

$$\frac{A \quad A \to B}{B}\ (\text{MP}) \ , \qquad \frac{c_1, \dots, c_n \in \mathsf{Cons} \quad C \text{ is an } \mathsf{OPAL(K)}\text{-axiom}}{[\sigma_1]c_1 : \dots : [\sigma_n]c_n : C}\ (\text{AN}) \ ,$$

where σ_i's are (possibly empty) finite sequences of formulas.

We sometimes use some of the same names for both axioms of $\mathsf{OPAL(K)}$ and axioms of $\mathsf{PAL(K)}$ because it will always be clear from the context which of the two is meant.

Besides $\mathsf{OPAL(K)}$, we also consider the deductive system $\mathsf{JPAL(K)}$, which does not assign any particular meaning to the two term operations $\Uparrow$ and $\Downarrow$.

Definition 12 ($\mathsf{JPAL(K)}$ Deductive System)**.** The *axioms of* $\mathsf{JPAL(K)}$ are the axioms of $\mathsf{OPAL(K)}$ where the two update axiom schemes are replaced by the single scheme

$$[\sigma]t : (A \to [A]B) \leftrightarrow [\sigma][A]t : B \ . \qquad \text{(update)}$$

The *deductive system* $\mathsf{JPAL(K)}$ is a Hilbert system that consists of the axioms of $\mathsf{JPAL(K)}$ and the rules (MP) and (AN), where the formula C in (AN) now stands for an axiom of $\mathsf{JPAL(K)}$.

We will use $\mathsf{OPAL(K)} \vdash A$ and $\mathsf{JPAL(K)} \vdash A$ to express that A is derivable in $\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$, respectively. If the deductive system does not matter, for instance when A is derivable in both of them, then we use $\vdash A$.

The following example gives some intuition as to how the deductive systems work and what their differences are.

Example 13. *For any $p \in \mathsf{Prop}$ and any $c_1, c_2 \in \mathsf{Cons}$, we have*

1. $\mathsf{OPAL(K)} \vdash [p] \Uparrow (c_1 \cdot c_2) : p$ *and*
2. $\mathsf{JPAL(K)} \vdash [p](c_1 \cdot c_2) : p$.

Proof. We use PR to denote the use of propositional reasoning. By AN for the tautology $([p]p \leftrightarrow p) \to (p \to [p]p)$ we have

$$\vdash c_1 : \big(([p]p \leftrightarrow p) \to (p \to [p]p)\big) \ . \tag{4}$$

By AN for the independence axiom $[p]p \leftrightarrow p$ we have

$$\vdash c_2 : ([p]p \leftrightarrow p) \ . \tag{5}$$

From (4) and (5) we obtain by the application axiom and PR

$$\vdash (c_1 \cdot c_2) : (p \to [p]p) \ . \tag{6}$$

In $\mathsf{OPAL(K)}$ we get from (6) by the update axiom $\Uparrow$ and PR

$$\mathsf{OPAL(K)} \vdash [p] \Uparrow (c_1 \cdot c_2) : p \ .$$

In $\mathsf{JPAL(K)}$ we get from (6) by the update axiom and PR

$$\mathsf{JPAL(K)} \vdash [p](c_1 \cdot c_2) : p \ . \qquad \square$$

We see that, independent of the truth value of an atomic proposition p, after p is announced, there is a reason to believe p. In $\mathsf{JPAL(K)}$ this reason is given by the term $c_1 \cdot c_2$. In $\mathsf{OPAL(K)}$ the term is $\Uparrow(c_1 \cdot c_2)$. In the latter case, the presence of $\Uparrow$ in the evidence term clearly signifies that this evidence for p is contingent on a prior public announcement. However, the exact content of such a public announcement, p in our case, is not recorded in the term. This design decision enables us to avoid the overcomplexification of the language and is similar to the introspection operation in the traditional justification logics: $!t$ is evidence for $t : A$ whenever t is evidence for A; however, the formula A is not recorded in the term $!t$.

Remark 14. The announcement-free fragment of $\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$ (that is the first three axiom schemes with $\sigma = \varepsilon$, rule MP, and rule AN, restricted to $c_1 : \cdots : c_n : C$) is the well-known justification logic J (see [5]).

The following lemma states a standard property of justification logics that holds for $\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$; it can be proved by an easy induction on the length of derivation.

Lemma 15 (Internalization)**.** *If $C_1, \ldots, C_n \vdash A$, then there is a term $t(y_1, \ldots, y_n)$ for fresh variables $y_1, \ldots, y_n$ such that*

$$y_1 : C_1, \ldots, y_n : C_n \vdash t(y_1, \ldots, y_n) : A \ .$$

Corollary 16 (Constructive Necessitation)**.** *For any formula A, if $\vdash A$, then there is a ground term t such that $\vdash t : A$.*

4 Semantics

We adapt the Kripke-style semantics for Justification Logic due to Fitting [9]. Our semantics uses Kripke models augmented by *evidence functions* that relate each world–term pair (w, t) to a set of formulas $\mathcal{E}(w, t)$ that the term t can justify at the world w.

Definition 17 (Frame)**.** A *frame* is a pair (W, R) that consists of a set $W \neq \emptyset$ of *(possible) worlds* and of an *accessibility relation* $R \subseteq W \times W$.

Definition 18 (Evidence Function)**.** A function $\mathcal{E}\colon W \times \mathsf{Tm} \to \mathcal{P}(\mathsf{Fml_J})$ is called *evidence function* if it satisfies the following closure conditions:

1. *Axioms:* if $c\colon A$ is derivable by the AN-rule, then $A \in \mathcal{E}(w, c)$ for any $w \in W$.
2. *Application:* if $(A \to B) \in \mathcal{E}(w, t)$ and $A \in \mathcal{E}(w, s)$, then $B \in \mathcal{E}(w, t\cdot s)$.
3. *Sum:* $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s+t)$ for any $s, t \in \mathsf{Tm}$ and any $w \in W$.

In a model of $\mathsf{OPAL(K)}$ or $\mathsf{JPAL(K)}$, there is an evidence function $\mathcal{E}^\sigma$ for each finite sequence σ of formulas. The idea is that the evidence function $\mathcal{E}^\sigma$ models the "evidential situation" that arises after the formulas in σ have been publicly announced.

Definition 19 (Model)**.** A *model* is a structure $\mathcal{M} = (W, R, \mathcal{E}, \nu)$, where (W, R) is a frame, $\nu : \mathsf{Prop} \to \mathcal{P}(W)$ is a *valuation*, and function $\mathcal{E}$ maps finite sequences σ of formulas to evidence functions $\mathcal{E}^\sigma$. An $\mathsf{OPAL(K)}$ *model* satisfies the following three conditions:

$$A \to [A]B \in \mathcal{E}^\sigma(w, t) \;\text{ implies }\; B \in \mathcal{E}^{\sigma,A}(w, \Uparrow t) \enspace , \tag{7}$$

$$B \in \mathcal{E}^{\sigma,A}(w, t) \;\text{ implies }\; A \to [A]B \in \mathcal{E}^\sigma(w, \Downarrow t) \enspace , \tag{8}$$

$$\mathcal{E}^{\sigma,A,B}(w, t) = \mathcal{E}^{\sigma,A\wedge[A]B}(w, t) \enspace . \tag{9}$$

A $\mathsf{JPAL(K)}$ *model* satisfies (9) and, instead of (7) and (8), the condition

$$A \to [A]B \in \mathcal{E}^\sigma(w, t) \;\text{ if and only if }\; B \in \mathcal{E}^{\sigma,A}(w, t) \enspace . \tag{10}$$

Conditions (7), (8), (9), and (10) correspond to the update axiom $\Uparrow$, the update axiom $\Downarrow$, the iteration axiom, and the update axiom of $\mathsf{JPAL(K)}$ respectively.

Remark 20. Our notion of model is non-empty. Simple sample models for $\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$ can be found in [8] and [6], respectively.

Definition 21 (Truth in a Model)**.** A ternary relation $\mathcal{M}, w \Vdash A$ for *formula A being satisfied at a world $w \in W$ in a model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$* is defined by induction on the structure of A:

- $\mathcal{M}, w \Vdash p$ if and only if $w \in \nu(p)$.
- Boolean connectives behave classically.
- $\mathcal{M}, w \Vdash t\!:\!A$ if and only if 1) $A \in \mathcal{E}^{\varepsilon}(w, t)$ and 2) $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $R(w, v)$.
- $\mathcal{M}, w \Vdash [A]B$ if and only if $\mathcal{M}_A, w \Vdash B$, where $\mathcal{M}_A = (W_A, R_A, \mathcal{E}_A, \nu_A)$ is defined as follows: $W_A := W$; $R_A := \{(s,t) \mid R(s,t) \text{ and } \mathcal{M}, t \Vdash A\}$; $(\mathcal{E}_A)^{\sigma} := \mathcal{E}^{A,\sigma}$; and $\nu_A := \nu$. Note that if $\mathcal{M}$ is an $\mathsf{OPAL(K)}$ model, then $\mathcal{M}_A$ satisfies conditions (7)–(9) from Def. 19 and hence also is an $\mathsf{OPAL(K)}$ model. Similarly, if $\mathcal{M}$ is a $\mathsf{JPAL(K)}$ model, then $\mathcal{M}_A$ satisfies conditions (9) and (10) and hence also is a $\mathsf{JPAL(K)}$ model.

We write $\mathcal{M} \Vdash A$ to mean that $\mathcal{M}, w \Vdash A$ for all $w \in W$. We say that formula A is $\mathsf{OPAL(K)}$ *valid*, written $\mathsf{OPAL(K)} \Vdash A$, to mean that $\mathcal{M} \Vdash A$ for all $\mathsf{OPAL(K)}$ models $\mathcal{M}$. Formula A is $\mathsf{JPAL(K)}$ *valid*, written $\mathsf{JPAL(K)} \Vdash A$, if $\mathcal{M} \Vdash A$ for all $\mathsf{JPAL(K)}$ models $\mathcal{M}$.

For a sequence $\tau = (A_1, \dots, A_n)$ of formulas we use $\mathcal{M}_\tau = (W_\tau, R_\tau, \mathcal{E}_\tau, \nu_\tau)$ to denote the model $(\cdots((\mathcal{M}_{A_1})_{A_2})\cdots)_{A_n}$. Note that $(\mathcal{E}_\tau)^{\sigma} = \mathcal{E}^{\tau,\sigma}$; in particular, $(\mathcal{E}_\tau)^{\varepsilon} = \mathcal{E}^{\tau}$.

To illustrate how the semantics works, we prove a semantic version of the result from Example 13.

Example 22. *For any $p \in \mathsf{Prop}$ and any $c_1, c_2 \in \mathsf{Cons}$, we have*

1. $\mathsf{OPAL(K)} \Vdash [p]\Uparrow(c_1 \cdot c_2)\!:\!p$ *and*
2. $\mathsf{JPAL(K)} \Vdash [p](c_1 \cdot c_2)\!:\!p$.

Proof. Let $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ be an arbitrary model and let $w \in W$. By Def. 18.1, we have $([p]p \leftrightarrow p) \to (p \to [p]p) \in \mathcal{E}^{\varepsilon}(w, c_1)$ and $([p]p \leftrightarrow p) \in \mathcal{E}^{\varepsilon}(w, c_2)$. Thus, $(p \to [p]p) \in \mathcal{E}^{\varepsilon}(w, c_1 \cdot c_2)$ by Def. 18.2.

Assume now that $\mathcal{M}$ is an $\mathsf{OPAL(K)}$ model. Then, by condition (7) from Def. 19, we have $p \in \mathcal{E}^{p}(w, \Uparrow(c_1 \cdot c_2))$. Since $R_p(w,v)$ implies $\mathcal{M}, v \Vdash p$, i.e., $v \in \nu(p) = \nu_p(p)$, we have $\mathcal{M}_p, w \Vdash \Uparrow(c_1 \cdot c_2)\!:\!p$ by Def. 21 and, hence, $\mathcal{M}, w \Vdash [p]\Uparrow(c_1 \cdot c_2)\!:\!p$.

Assume that $\mathcal{M}$ is a $\mathsf{JPAL(K)}$ model. Then, by condition (10) from Def. 19, we have $p \in \mathcal{E}^{p}(w, c_1 \cdot c_2)$. As above we then find $\mathcal{M}_p, w \Vdash (c_1 \cdot c_2)\!:\!p$ and $\mathcal{M}, w \Vdash [p](c_1 \cdot c_2)\!:\!p$. $\square$

5 Soundness and Completeness

Lemma 23 (Soundness)**.** *For all formulas $A \in \mathsf{Fml_J}$, we have*

1. $\mathsf{OPAL(K)} \vdash A$ *implies* A *is* $\mathsf{OPAL(K)}$ *valid,*
2. $\mathsf{JPAL(K)} \vdash A$ *implies* A *is* $\mathsf{JPAL(K)}$ *valid.*

Proof. As usual the proof is by induction on the length of the derivation of A. We only show the cases concerning the axioms about announcements.

1. Independence ($\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$). $\mathcal{M}, w \Vdash [\sigma]p$ iff $\mathcal{M}_\sigma, w \Vdash p$ iff $w \in \nu_\sigma(p)$ iff $w \in \nu(p)$ iff $\mathcal{M}, w \Vdash p$.

2. Normality ($\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$). $\mathcal{M}, w \Vdash [\sigma](B \to C)$ iff $\mathcal{M}_\sigma, w \Vdash B \to C$ iff $\mathcal{M}_\sigma, w \nVdash B$ or $\mathcal{M}_\sigma, w \Vdash C$ iff $\mathcal{M}, w \nVdash [\sigma]B$ or $\mathcal{M}, w \Vdash [\sigma]C$ iff $\mathcal{M}, w \Vdash [\sigma]B \to [\sigma]C$.

3. Functionality ($\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$). $\mathcal{M}, w \Vdash [\sigma]\neg B$ iff $\mathcal{M}_\sigma, w \Vdash \neg B$ iff $\mathcal{M}_\sigma, w \nVdash B$ iff $\mathcal{M}, w \nVdash [\sigma]B$ iff $\mathcal{M}, w \Vdash \neg[\sigma]B$.

4. Update ($\mathsf{JPAL(K)}$). $\mathcal{M}, w \Vdash [\sigma]t : (A \to [A]B)$ is equivalent to the conjunction of
$$A \to [A]B \in \mathcal{E}^\sigma(w, t) \tag{11}$$
and
$$\mathcal{M}_\sigma, v \Vdash A \to [A]B \text{ for all } v \text{ with } R_\sigma(w, v) \ . \tag{12}$$
By the condition (10) on $\mathcal{E}$ from Def. 19, we obtain that (11) if and only if
$$B \in \mathcal{E}^{\sigma,A}(w, t) \ . \tag{13}$$
Moreover, (12) is equivalent to
$$\mathcal{M}_\sigma, v \Vdash A \text{ implies } \mathcal{M}_\sigma, v \Vdash [A]B \text{ for all } v \text{ with } R_\sigma(w, v) \ .$$
This is equivalent to
$$\mathcal{M}_\sigma, v \Vdash A \text{ implies } \mathcal{M}_{\sigma,A}, v \Vdash B \text{ for all } v \text{ with } R_\sigma(w, v) \ ,$$
which, in turn, is equivalent to
$$\mathcal{M}_{\sigma,A}, v \Vdash B \text{ for all } v \text{ with } R_{\sigma,A}(w, v) \ .$$
The conjunction of this and (13) is equivalent to
$$\mathcal{M}_{\sigma,A}, w \Vdash t : B \ ,$$
or, equivalently,
$$\mathcal{M}, w \Vdash [\sigma][A]t : B \ .$$

5. Update $\Uparrow$ ($\mathsf{OPAL(K)}$). This case is similar to the $\Rightarrow$ direction of the update case for $\mathsf{JPAL(K)}$.

6. Update $\Downarrow$ ($\mathsf{OPAL(K)}$). This case is similar to the $\Leftarrow$ direction of the update case for $\mathsf{JPAL(K)}$.

7. Iteration ($\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$). First we show that

$$R_{\sigma,A,B} = R_{\sigma,A\wedge[A]B} \ . \tag{14}$$

$R_{\sigma,A,B}(u,v)$ is equivalent to

$$R_\sigma(u,v) \text{ and } \mathcal{M}_\sigma, v \Vdash A \text{ and } \mathcal{M}_{\sigma,A}, v \Vdash B \ .$$

This is equivalent to

$$R_\sigma(u,v) \text{ and } \mathcal{M}_\sigma, v \Vdash A \wedge [A]B \ ,$$

which, in turn, is equivalent to $R_{\sigma,A\wedge[A]B}(u,v)$ and thus (14) is established.

The case for iteration is now as follows:

$$\mathcal{M}, w \Vdash [\sigma][A][B]C$$

if and only if

$$\mathcal{M}_{\sigma,A,B}, w \Vdash C \ .$$

By condition (9) on $\mathcal{E}$ from Def. 19 and by (14), this is equivalent to

$$\mathcal{M}_{\sigma,A\wedge[A]B}, w \Vdash C$$

which, in turn, is equivalent to

$$\mathcal{M}, w \Vdash [\sigma][A \wedge [A]B]C \ . \qquad \square$$

The traditional modal logic reduction approach (see Remark 7) to establishing completeness is not possible in the presence of justifications since the replacement property does not hold in Justification Logic (see [10, Sect. 6] for a detailed discussion of the replacement property in Justification Logic). That means, in particular, that $\vdash A \leftrightarrow B$ does not imply $\vdash t\!:\!A \leftrightarrow t\!:\!B$, which would be an essential step in the proof of a justification-analog of Lemma 6. Thus, it is not possible to transfer the completeness of J (see [9, 15]) to $\mathsf{OPAL(K)}$ or $\mathsf{JPAL(K)}$. We will, instead, provide a canonical model construction to prove the completeness of $\mathsf{OPAL(K)}$ and $\mathsf{JPAL(K)}$. In the following we let S stand for either $\mathsf{OPAL(K)}$ or $\mathsf{JPAL(K)}$.

Definition 24 (Maximal S-Consistent Sets)**.** A set Φ of $\mathsf{Fml_J}$-formulas is called S-*consistent* if there is a formula that cannot be derived from Φ in S. A set Φ is called *maximal* S-*consistent* if it is consistent but has no consistent proper extensions.

It can be easily shown that maximal S-consistent sets contain all axioms of S and are closed under modus ponens and axiom necessitation.

Definition 25 (Canonical S Model)**.** We define the *canonical* S *model* $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ as follows:

1. $W := \{w \subseteq \mathsf{Fml_J} \mid w \text{ is a maximal } \mathsf{S}\text{-consistent set}\}$,
2. $R(w, v)$ if and only if for all finite sequences σ and all $t \in \mathsf{Tm}$, we have $[\sigma]t : A \in w$ implies $[\sigma]A \in v$,
3. $\mathcal{E}^\sigma(w, t) := \{A \in \mathsf{Fml_J} \mid [\sigma]t : A \in w\}$,
4. $\nu(p) := \{w \in W \mid p \in w\}$.

To establish completeness, we need to know that the canonical model is a model.

Lemma 26 (Correctness of the Canonical Model)**.**

1. The canonical $\mathsf{OPAL(K)}$ *model is an* $\mathsf{OPAL(K)}$ *model.*

2. The canonical $\mathsf{JPAL(K)}$ *model is a* $\mathsf{JPAL(K)}$ *model.*

Proof. First, we observe that the set W is non-empty: by Remark 20, there exists a model and the set of all formulas that are true at some world of the model is maximally consistent. We next show that $\mathcal{E}^\sigma$ is an evidence function for each σ.

- Axioms. For any $c : A$ derivable by AN, $[\sigma]c : A$ is also derivable for any σ. Hence, $[\sigma]c : A \in w$ and $A \in \mathcal{E}^\sigma(w, c)$.
- Application. Assume $A \to B \in \mathcal{E}^\sigma(w, t)$ and $A \in \mathcal{E}^\sigma(w, s)$. We then have $[\sigma]t : (A \to B) \in w$ and $[\sigma]s : A \in w$. By the application and normality axioms, we get $[\sigma](t \cdot s) : B \in w$. Thus, $B \in \mathcal{E}^\sigma(w, t \cdot s)$.
- Sum. This is similar to the previous case.

For a canonical $\mathsf{OPAL}(\mathsf{K})$ model $\mathcal{M}$ it remains to be shown that $\mathcal{M}$ satisfies conditions (7), (8), and (9) from Def. 19. Let us show condition (7). We have $A \to [A]B \in \mathcal{E}^\sigma(w,t)$ if and only if $[\sigma]t:(A \to [A]B) \in w$. By the update axiom $\Uparrow$, the latter implies $[\sigma][A]\Uparrow t:B \in w$, which is equivalent to $B \in \mathcal{E}^{\sigma,A}(w, \Uparrow t)$. Conditions (8) and (9) are shown similarly using the update axiom $\Downarrow$ and the iteration axiom respectively.

For a canonical $\mathsf{JPAL}(\mathsf{K})$ model $\mathcal{M}$ it remains to be shown that $\mathcal{M}$ satisfies conditions (10) and (9) from Def. 19, which is similar to the proof of the conditions for $\mathsf{OPAL}(\mathsf{K})$. $\square$

Remark 27. The canonical $\mathsf{OPAL}(\mathsf{K})$ model and the canonical $\mathsf{JPAL}(\mathsf{K})$ model are both degenerate, i.e., the canonical model consists of isolated irreflexive worlds.

Proof. $\bot \to F$ is an axiom for any F. In particular, $\vdash \bot \to [\bot]F$ is an axiom for any F. By AN, for any F, we have $\vdash c:(\bot \to [\bot]F)$, where c is a constant. By the update axiom and the update axiom $\Uparrow$ respectively, for any formula F, we have $\vdash [\bot]s:F$ for some ground term s. Hence, for any F, the formula $[\bot]s:F$ is contained in each maximal consistent set, that is in each world of the canonical model. Let w be such a world and assume towards a contradiction that v is accessible from w. We then have by Def. 25 that $[\bot]F \in v$ for any F. In particular, both $[\bot]F \in v$ and $[\bot]\neg F \in v$. By the functionality axiom, the latter implies $\neg[\bot]F \in v$, which contradicts $[\bot]F \in v$ since v is consistent. Thus, there cannot be a world v that is accessible from w. $\square$

Definition 28 (Rank)**.** The *rank* $\mathsf{rk}(A)$ of a formula A is defined as follows:

1. $\mathsf{rk}(p) := 1$ for each $p \in \mathsf{Prop}$
2. $\mathsf{rk}(\neg A) := \mathsf{rk}(A) + 1$
3. $\mathsf{rk}(A \to B) := \max(\mathsf{rk}(A), \mathsf{rk}(B)) + 1$
4. $\mathsf{rk}(t:A) := \mathsf{rk}(A) + 1$
5. $\mathsf{rk}([A]B) := (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B)$

Lemma 29 (Reductions Reduce Rank)**.** *For all formulas A, B, C and all terms s, t, we have the following:*

1. $\mathsf{rk}(A) > \mathsf{rk}(B)$ *if B is a proper subformula of A*
2. $\mathsf{rk}([A]\neg B) > \mathsf{rk}(\neg[A]B)$

3. $\mathsf{rk}([A](B \to C)) > \mathsf{rk}([A]B \to [A]C)$

4. $\mathsf{rk}([A]s : B) > \mathsf{rk}(t : (A \to [A]B))$

5. $\mathsf{rk}([A][B]C) > \mathsf{rk}([A \wedge [A]B]C)$

Proof. Let us only show the last two cases. First, case 4.

$$\begin{aligned}\mathsf{rk}([A]s : B) &= (2 + \mathsf{rk}(A)) \cdot (\mathsf{rk}(B) + 1)\\ &= (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B) + 2 + \mathsf{rk}(A)\\ &> (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B) + 1 + 1\\ &= \mathsf{rk}(t : (A \to [A]B)) \ .\end{aligned}$$

And now case 5.

$$\begin{aligned}\mathsf{rk}([A][B]C) &= (2 + \mathsf{rk}(A)) \cdot (2 + \mathsf{rk}(B)) \cdot \mathsf{rk}(C)\\ &= (4 + 2\mathsf{rk}(A) + 2\mathsf{rk}(B) + \mathsf{rk}(A)\mathsf{rk}(B)) \cdot \mathsf{rk}(C)\\ &\geq (6 + 2\mathsf{rk}(B) + \mathsf{rk}(A)\mathsf{rk}(B)) \cdot \mathsf{rk}(C)\\ &> (2 + 3 + (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B)) \cdot \mathsf{rk}(C)\\ &= (2 + \mathsf{rk}(\neg(A \to \neg[A]B))) \cdot \mathsf{rk}(C)\\ &= \mathsf{rk}([A \wedge [A]B]C) \ .\end{aligned}$$ □

Lemma 30 (Truth Lemma)**.** *Let $\mathcal{M}$ be the canonical* $\mathsf{OPAL(K)}$ *model or the canonical* $\mathsf{JPAL(K)}$ *model. For all formulas D and all worlds w in $\mathcal{M}$, we have $D \in w$ if and only if $\mathcal{M}, w \Vdash D$.*

Proof. Proof by induction on $\mathsf{rk}(D)$ and a case distinction on the structure of D. Let us only show the cases where D is of the form $[A]B$. The other cases are standard and follow easily from the maximal consistency of w and the definition of the canonical model.

1. $D = [A]p$. Suppose $[A]p \in w$. By the independence axiom, this is equivalent to $p \in w$. By the induction hypothesis, this is equivalent to $\mathcal{M}, w \Vdash p$, which is equivalent to $\mathcal{M}, w \Vdash [A]p$ by the soundness of the independence axiom.

2. $D = [A]\neg B$. Suppose $[A]\neg B \in w$. By the functionality axiom, this is equivalent to $\neg[A]B \in w$. By the induction hypothesis, this is equivalent to $\mathcal{M}, w \Vdash \neg[A]B$, which is equivalent to $\mathcal{M}, w \Vdash [A]\neg B$ by the soundness of the functionality axiom.

3. $D = [A](B \to C)$ is shown similarly using the normality axiom.

4. $D = [A]t : B$ and $\mathcal{M}$ is the canonical $\mathsf{OPAL(K)}$ model. Suppose $[A]t : B \in w$. By the update axiom $\Downarrow$, we then have $\Downarrow t : (A \to [A]B) \in w$. By the induction hypothesis, this is equivalent to $\mathcal{M}, w \Vdash \Downarrow t : (A \to [A]B)$, which implies, in particular, that $\mathcal{M}, v \Vdash A \to [A]B$ whenever $R(w, v)$. Equivalently, $\mathcal{M}, v \Vdash [A]B$ whenever $R(w, v)$ and $\mathcal{M}, v \Vdash A$. Equivalently, $\mathcal{M}_A, v \Vdash B$ whenever $R_A(w, v)$. In addition, by the definition of $\mathcal{E}$, we have $B \in \mathcal{E}^A(w, t)$. To summarize, we have $\mathcal{M}_A, w \Vdash t : B$. In other words, $\mathcal{M}, w \Vdash [A]t : B$.

   Suppose $[A]t : B \notin w$. By the definition of $\mathcal{E}$, we have $B \notin \mathcal{E}^A(w, t)$. Hence, $\mathcal{M}_A, w \nVdash t : B$. In other words, $\mathcal{M}, w \nVdash [A]t : B$.

5. $D = [A]t : B$ and $\mathcal{M}$ is the canonical $\mathsf{JPAL(K)}$ model. Suppose $[A]t : B \in w$. By the update axiom this is equivalent to $t : (A \to [A]B) \in w$. By the induction hypothesis this is equivalent to $\mathcal{M}, w \Vdash t : (A \to [A]B)$, which by the soundness of the update axiom is equivalent to $\mathcal{M}, w \Vdash [A]t : B$.

6. $D = [A][B]C$. Suppose $[A][B]C \in w$. By the iteration axiom, this is equivalent to $[A \wedge [A]B]C \in w$. By the induction hypothesis, this is equivalent to $\mathcal{M}, w \Vdash [A \wedge [A]B]C$, which, by the soundness of the iteration axiom, is equivalent to $\mathcal{M}, w \Vdash [A][B]C$. □

As usual, the Truth Lemma implies completeness, which, as a corollary, yields announcement necessitation.

Theorem 31 (Completeness)**.** *For all formulas $A \in \mathsf{Fml_J}$, we have*

1. $\mathsf{OPAL(K)} \vdash A$ *if and only if A is* $\mathsf{OPAL(K)}$ *valid,*

2. $\mathsf{JPAL(K)} \vdash A$ *if and only if A is* $\mathsf{JPAL(K)}$ *valid.*

Proof. Soundness was already shown in Lemma 23. For completeness of $\mathsf{OPAL(K)}$, consider the canonical $\mathsf{OPAL(K)}$ model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ and assume that $\mathsf{OPAL(K)} \nvdash A$. Then $\{\neg A\}$ is consistent and, hence, contained in some maximal consistent set $w \in W$. By Lemma 30, it follows that $\mathcal{M}, w \Vdash \neg A$ and, hence, that $\mathcal{M}, w \nVdash A$. Since $\mathcal{M}$ is an $\mathsf{OPAL(K)}$ model (Lemma 26), we have shown that $\mathsf{OPAL(K)} \nvdash A$ implies $\mathsf{OPAL(K)} \nVdash A$. Completeness of $\mathsf{OPAL(K)}$ follows by contraposition. Completeness of $\mathsf{JPAL(K)}$ is established similarly. □

Corollary 32 (Announcement Necessitation)**.** *Announcement necessitation is admissible: that is, for all formulas $A, B \in \mathsf{Fml_J}$, we have*

1. $\mathsf{OPAL(K)} \vdash A$ *implies* $\mathsf{OPAL(K)} \vdash [B]A$*,*

2. $\mathsf{JPAL(K)} \vdash A$ *implies* $\mathsf{JPAL(K)} \vdash [B]A$.

Proof. Assume $\mathsf{OPAL(K)} \vdash A$. By soundness, $\mathsf{OPAL(K)} \Vdash A$. Therefore, $\mathcal{M} \Vdash A$ for all $\mathsf{OPAL(K)}$ models $\mathcal{M}$. In particular, $\mathcal{M}_B, w \Vdash A$ for all $\mathsf{OPAL(K)}$ models of the form $\mathcal{M}_B$ and worlds w in them. Thus, we obtain $\mathcal{M}, w \Vdash [B]A$ for all $\mathcal{M}, w$. By completeness, we conclude $\mathsf{OPAL(K)} \vdash [B]A$. The case for $\mathsf{JPAL(K)}$ is shown similarly. □

6 Forgetful Projection and Realization

This section deals with the relationship between $\mathsf{PAL(K)}$ and dynamic justification logics. We show that for any theorem of either $\mathsf{OPAL(K)}$ or $\mathsf{JPAL(K)}$, its forgetful projection, which is obtained by replacing each term with $\Box$, is a theorem of $\mathsf{PAL(K)}$.

Definition 33 (Forgetful Projection)**.** The mapping $\circ : \mathsf{Fml_J} \to \mathsf{Fml}_{\Box,[\cdot]}$ is defined as follows:

$$
\begin{array}{llll}
p^\circ & = p \text{ for all } p \in \mathsf{Prop}, & \circ \text{ commutes with connectives } \neg \text{ and } \to \ , & \\
(t : A)^\circ & = \Box A^\circ & ([A]B)^\circ & = [A^\circ]B^\circ \ .
\end{array}
$$

For a sequence $\sigma = (A_1, \dots, A_n)$ of $\mathsf{Fml_J}$-formulas, we define σ° to be the sequence $({A_1}^\circ, \dots, {A_n}^\circ)$ of $\mathsf{Fml}_{\Box,[\cdot]}$-formulas. In particular, $\varepsilon^\circ := \varepsilon$.

Theorem 34 (Forgetful Projection of $\mathsf{JPAL(K)}$ and $\mathsf{OPAL(K)}$)**.** *For all formulas $A \in \mathsf{Fml_J}$,*

$$
\begin{array}{ccc}
\mathsf{JPAL(K)} \vdash A & \Longrightarrow & \mathsf{PAL(K)} \vdash A^\circ \ , \\
\mathsf{OPAL(K)} \vdash A & \Longrightarrow & \mathsf{PAL(K)} \vdash A^\circ \ .
\end{array}
$$

Proof. We use induction on a derivation in $\mathsf{JPAL(K)}$, respectively in $\mathsf{OPAL(K)}$. We need to show that the statement holds for all the axioms of $\mathsf{JPAL(K)}$, respectively of $\mathsf{OPAL(K)}$, for all instances of the rule AN, and is preserved by the rule MP, the two rules being common for the two logics. To simplify dealing with axioms, we first note that for each sequence σ of $\mathsf{Fml_J}$-formulas, there exists a formula $U_\sigma \in \mathsf{Fml}_{\Box,[\cdot]}$ such that $\mathsf{PAL(K)} \vdash [\sigma^\circ]D \leftrightarrow [U_\sigma]D$ for any formula $D \in \mathsf{Fml}_{\Box,[\cdot]}$. This is trivial if $|\sigma| = 1$ and follows from the modal iteration axiom if $|\sigma| > 1$. If $\sigma = \varepsilon$, $U_\varepsilon := p \vee \neg p$ for some proposition p.

Thus, to show that forgetful projections of all axioms, of $\mathsf{JPAL(K)}$ and $\mathsf{OPAL(K)}$, are derivable in $\mathsf{PAL(K)}$, it is sufficient to consider these projections with $[U_\sigma]$ substituted for $[\sigma^\circ]$. Most of the axioms are actually common between the two logics, which makes the proof shorter. If A is a propositional tautology, so is A°. Therefore, $[U_\sigma]A^\circ$ can be derived in $\mathsf{PAL(K)}$ by using the announcement necessitation rule, which is known to be admissible in $\mathsf{PAL(K)}$ by Lemma 3. Similarly, the application and sum axioms have a form $[\sigma]C$ where C° is a theorem of $\mathsf{PAL(K)}$: in case of the application axiom, $C^\circ = \Box(A^\circ \to B^\circ) \to (\Box A^\circ \to \Box B^\circ)$, it is an axiom of K; in case of the sum axiom, $C^\circ = \Box A^\circ \to \Box A^\circ$, it is a propositional tautology. So again $[U_\sigma]C^\circ$ is derivable via announcement necessitation. For the independence, normality, and functionality axioms, $[U_\sigma]p \leftrightarrow p$, $[U_\sigma](B^\circ \to C^\circ) \leftrightarrow ([U_\sigma]B^\circ \to [U_\sigma]C^\circ)$, and $[U_\sigma]\neg B^\circ \leftrightarrow \neg[U_\sigma]B^\circ$ are instances of the modal independence, normality, and functionality axioms, respectively. Finally, the remaining axioms of $\mathsf{JPAL(K)}$ have the form $[\sigma]D \leftrightarrow [\sigma]E$ where $D^\circ \leftrightarrow E^\circ$ is an instance of the corresponding modal axiom: $\Box(A^\circ \to [A^\circ]B^\circ) \leftrightarrow [A^\circ]\Box B^\circ$ and $[A^\circ][B^\circ]C^\circ \leftrightarrow [A^\circ \wedge [A^\circ]B^\circ]C^\circ$, respectively. The update $\Uparrow$ and update $\Downarrow$ axioms of $\mathsf{OPAL(K)}$ have a form $[\sigma]D \to [\sigma]E$ where $D^\circ \to E^\circ$ is one direction of the equivalence in the modal update axiom. It remains to note that $\dfrac{F \to G}{[U]F \to [U]G}$ is known to be admissible in $\mathsf{PAL(K)}$.

The forgetful projection of the AN rule has a form $[\sigma_1{}^\circ]\Box \dots [\sigma_n{}^\circ]\Box C^\circ$ where the derivability of C° in $\mathsf{PAL(K)}$ has just been demonstrated. Therefore, it is sufficient to use the modal necessitation rule n times and the announcement necessitation rule $|\sigma_1| + \cdots + |\sigma_n|$ times to get the desired result.

Finally, if B is obtained by MP from $A \to B$ and A, by induction hypothesis, $A^\circ \to B^\circ$ and A° are theorems of $\mathsf{PAL(K)}$. Hence, so is B°. $\Box$

A much more difficult question is whether a dynamic justification logic, such as $\mathsf{JPAL(K)}$ or $\mathsf{OPAL(K)}$, can realize $\mathsf{PAL(K)}$: that is, whether for any theorem A of $\mathsf{PAL(K)}$, it is possible to replace each $\Box$ in A with some term such that the resulting formula is a dynamic justification validity.

In the remainder of this paper we present the first realization technique for dynamic justification logics and establish a partial realization result for $\mathsf{JPAL(K)}$: it can realize formulas A that do not contain $\Box$ operators within announcements. Our main idea is to reduce realization of $\mathsf{PAL(K)}$ to realization of K. In our proof, we rely on notions and techniques introduced by Fitting [10].

Definition 35 (Substitution)**.** A *substitution* is a mapping from variables to terms. If A is a formula and σ is a substitution, we write $A\sigma$ to denote the result of simultaneously replacing each variable x in A with the term $x\sigma$.

The following lemma is standard in justification logics and can be proved by a simple induction on the derivation of A.

Lemma 36 (Substitution Lemma)**.** *For every formula A of* $\mathsf{Fml_J}$ *and every substitution σ,*

$$\mathsf{JPAL(K)} \vdash A \textit{ implies } \mathsf{JPAL(K)} \vdash A\sigma,$$
$$\mathsf{OPAL(K)} \vdash A \textit{ implies } \mathsf{OPAL(K)} \vdash A\sigma.$$

In most justification logics, in addition to this substitution of proof terms for proof variables, the substitution of formulas for propositions is also possible (see [2]). However, the latter type of substitution typically fails in logics with public announcements, as it does in both $\mathsf{JPAL(K)}$ and $\mathsf{OPAL(K)}$.

Definition 37 (Annotations)**.** An *annotated formula* is an $\mathsf{Fml}_{\Box,[\cdot]}$-formula in which each modal operator is annotated by a natural number. An annotated formula is *properly annotated* if modalities in negative positions are annotated with even numbers, modalities in positive positions are annotated with odd numbers, and no index i annotates two modality occurrences. Positions within an announcement $[A]$ are considered neither positive nor negative: i.e., the parity of indices within announcements in properly annotated formulas is not regulated. If A' is the result of replacing all indexed modal operators $\Box_i$ with $\Box$ in a (properly) annotated formula A, then A is called a *(properly) annotated version of* A'.

Definition 38 (Realization Function)**.** A *realization function* r is a mapping from natural numbers to terms such that $r(2i) = x_i$, where $x_1, x_2, \ldots$ is a fixed enumeration of all variables. For a realization function r and an annotated formula A, $r(A)$ denotes the result of replacing each indexed modal operator $\Box_i$ in A with the term $r(i)$. For instance, $r(\Box_i B) = r(i) : r(B)$. A realization function r is called *non-self-referential on variables over* A if, for each subformula $\Box_{2i} B$ of A, the variable $x_i = r(2i)$ does not occur in $r(B)$.

The following realization result for the logic K is due to Brezhnev [5]; the additional result about non-self-referentiality on variables follows from the stronger statement that K can be realized without any self-referential cycles of arbitrary terms, proved in [14].

Theorem 39 (Realization for K)**.** *If A' is a theorem of* K*, then for any properly annotated version A of A', there is a realization function r that is non-self-referential on variables over A and such that $r(A)$ is provable in* J*. Clearly, $(r(A))^\circ = A'$.*

In order to formulate the replacement theorem for $\mathsf{JPAL(K)}$, a technical result necessary for demonstrating the partial realization theorem for $\mathsf{JPAL(K)}$, we use the following standard convention: whenever $D(q)$ and A are formulas in the same language, $D(A)$ is the result of replacing all occurrences of the proposition q in $D(q)$ with A. In most cases, q has only one occurrence in $D(q)$ that is not within announcements.

For the rest of this section, we consider only formulas $A \in \mathsf{Fml}_{\Box,[\cdot]}$ and their annotated versions that do not contain modal operators within announcements: i.e., if $[B]C$ is a subformula of A, then B does not contain modal operators.

We will use a theorem that was first proved by Fitting [10] for replacement in positive positions in LP. We use his method in a richer language and for a different logic and also use replacement in both positive and negative positions. Thus, in the interests of self-containment, we give a more general formulation that we need and prove it. The proof is an adaptation of Fitting's proof from [10].

Theorem 40 (Restricted Realization Modification for $\mathsf{JPAL(K)}$)**.** *Assume the following:*

H-1. A proposition p has exactly[1] *one occurrence in a properly annotated formula $X(p)$ that is outside of announcements. $X(A)$ and $X(B)$ are properly annotated formulas with no modalities within announcements.*

H-2. r_1 is a realization function, non-self-referential on variables over $X(A)$.

H-3. If p occurs positively in $X(p)$, $\mathsf{JPAL(K)} \vdash r_1(A) \to r_1(B)$.
If p occurs negatively in $X(p)$, $\mathsf{JPAL(K)} \vdash r_1(B) \to r_1(A)$.

Then for each subformula $\varphi(p)$ of $X(p)$ that occurs outside of announcements, there is some realization/substitution pair $\langle r_\varphi, \sigma_\varphi\rangle$ such that:

C-1. $\mathsf{JPAL(K)} \vdash r_1(\varphi(A))\sigma_\varphi \to r_\varphi(\varphi(B))$ if $\varphi(p)$ occurs positively in $X(p)$.
$\mathsf{JPAL(K)} \vdash r_\varphi(\varphi(B)) \to r_1(\varphi(A))\sigma_\varphi$ if $\varphi(p)$ occurs negatively in $X(p)$.

C-2. σ_φ lives on input positions in $\varphi(p)$, i.e., $x_i\sigma_\varphi = x_i$ if $\Box_{2i}$ does not occur in $\varphi(p)$;

C-3. σ_φ meets the no new variable condition, i.e., the only variable that may occur in $x\sigma_\varphi$ is x;

C-4. If r_1 is non-self-referential on variables over $X(B)$, then r_φ is also non-self-referential on variables over $X(B)$.

[1]While the proof of this theorem does not depend on whether p actually occurs in $X(p)$, the formulation of H-1 is simpler when it does.

Proof. Assume that hypotheses H-1, H-2, and H-3 hold. We proceed by induction on the complexity of the subformula $\varphi(p)$. Call a subformula occurrence $\varphi(p)$ of $X(p)$ *good* provided there is some $\langle r_\varphi, \sigma_\varphi \rangle$ such that C-1, C-2, C-3, and C-4 hold; we also say $\langle r_\varphi, \sigma_\varphi \rangle$ is a *witness* to the goodness of $\varphi(p)$.[2] We show that every subformula occurrence of $X(p)$ that is outside of announcements is good.

Let $\varphi(p)$ be a subformula occurrence of $X(p)$ that is outside of announcements. Then this occurrence must be either positive or negative in $X(p)$. Assume as an induction hypothesis, that all its proper subformula occurrences are good; note that they must also be outside of announcements. We show $\varphi(p)$ itself is good. There are several cases to consider.

Base case. If $\varphi(p)$ is atomic, set $r_\varphi := r_1$ and σ_φ to be the identity substitution.

C-2 and C-3. The identity substitution lives on input positions in any formula and meets the no new variable condition.

C-4. If r_1 is non-self-referential on variables over $X(B)$, so is $r_\varphi = r_1$.

C-1. Here further subcases have to be considered:

C-1. Base subcase 1. $\varphi(p) = q$, where q is a proposition different from p. Whether this occurrence of q is positive or negative in $X(p)$,

$$r_1(\varphi(A))\sigma_\varphi \to r_\varphi(\varphi(B)) = r_\varphi(\varphi(B)) \to r_1(\varphi(A))\sigma_\varphi = q \to q \enspace ,$$

which is clearly derivable in $\mathsf{JPAL}(\mathsf{K})$.

C-1. Base subcase 2. $\varphi(p) = p$. This subformula occurs in $X(p)$ exactly once. If it occurs positively, C-1 requires $\mathsf{JPAL}(\mathsf{K}) \vdash r_1(\varphi(A))\sigma_\varphi \to r_\varphi(\varphi(B))$ or $\mathsf{JPAL}(\mathsf{K}) \vdash r_1(A) \to r_1(B)$, which follows from H-3 for the positively occurring p.

If p occurs negatively, C-1 requires $\mathsf{JPAL}(\mathsf{K}) \vdash r_\varphi(\varphi(B)) \to r_1(\varphi(A))\sigma_\varphi$ or $\mathsf{JPAL}(\mathsf{K}) \vdash r_1(B) \to r_1(A)$, which follows from H-3 for the negatively occurring p.

Negation case. If $\varphi(p) = \neg\theta(p)$ and $\langle r_\theta, \sigma_\theta \rangle$ has been constructed, set $r_\varphi := r_\theta$ and $\sigma_\varphi := \sigma_\theta$. We show that $\langle r_\varphi, \sigma_\varphi \rangle$ witnesses the goodness of an occurrence of $\neg\theta(p)$ in $X(p)$ under the assumption that $\langle r_\theta, \sigma_\theta \rangle$ witnesses the goodness of the corresponding occurrence of $\theta(p)$.

[2] Despite $X(p)$ being properly annotated, a subformula $\varphi(p)$ may have several occurrences in $X(p)$, including occurrences of opposite polarities, if $\varphi(p)$ contains no modalities. While r_φ and σ_φ are the same for all occurrences of $\varphi(p)$, independent of their polarities, the statement of C-1 does depend on the polarity of of the occurrence. Hence, we define r_φ and σ_φ for subformulas, but the goodness must be determined for each occurrence of a subformula separately.

C-2, C-3, and C-4 for $\varphi(p)$ easily follow from C-2, C-3, and C-4 for $\theta(p)$ because $r_\varphi = r_\theta$ and $\sigma_\varphi = \sigma_\theta$. For C-2, it is sufficient to note that any $\Box_{2i}$ occurring in $\theta(p)$ also occurs in $\neg\theta(p)$.

C-1. The statement that needs to be demonstrated depends on the polarity of this occurrence of $\neg\theta(p)$. We only show one of the two cases since the other is analogous. Let this occurrence be positive. Then the corresponding occurrence of $\theta(p)$ is negative and $\mathsf{JPAL}(\mathsf{K}) \vdash r_\theta(\theta(B)) \to r_1(\theta(A))\sigma_\theta$ by C-1 for $\theta(p)$. By contraposition, $\mathsf{JPAL}(\mathsf{K}) \vdash r_1(\neg\theta(A))\sigma_\varphi \to r_\varphi(\neg\theta(B))$, which is C-1 for the positively occurring $\neg\theta(p)$.

Implication case. If $\varphi(p) = \theta(p) \to \eta(p)$ and $\langle r_\theta, \sigma_\theta\rangle$ and $\langle r_\eta, \sigma_\eta\rangle$ have been constructed, set $\sigma_\varphi := \sigma_\theta\sigma_\eta$ and

$$r_\varphi(n) := \begin{cases} r_\theta(n)\sigma_\eta & \text{if } \Box_n \text{ occurs in } \theta(B) \\ r_\eta(n)\sigma_\theta & \text{if } \Box_n \text{ occurs in } \eta(B) \\ r_1(n) & \text{otherwise.} \end{cases}.$$

This r_φ is well-defined, i.e., $\Box_n$ cannot occur in both $\theta(B)$ and $\eta(B)$, since $\theta(B) \to \eta(B)$ is a subformula of $X(B)$, which is properly annotated by H-1. We show that $\langle r_\varphi, \sigma_\varphi\rangle$ witnesses the goodness of an occurrence of $\theta(p) \to \eta(p)$ in $X(p)$ whenever $\langle r_\theta, \sigma_\theta\rangle$ and $\langle r_\eta, \sigma_\eta\rangle$ witness the goodness of the corresponding occurrences of $\theta(p)$ and $\eta(p)$.

It is also necessary to show that our r_φ is a realization function, i.e., $r_\varphi(2i) = x_i$. Since $r_\theta(2i) = r_\eta(2i) = r_1(2i) = x_i$, we only need to show that neither σ_η, when $\Box_{2i}$ occurs in $\theta(B)$, nor σ_θ, when $\Box_{2i}$ occurs in $\eta(B)$, changes x_i. Given the symmetry of the two situations, we only consider the former, when $\Box_{2i}$ occurs in $\theta(B)$. Since $\theta(B) \to \eta(B)$ is a subformula of $X(B)$, which is properly annotated by H-1, $\theta(B)$ can share no annotations with $\eta(B)$, and even less so with $\eta(p)$. Thus, x_i is not in input position in $\eta(p)$ and is not changed by σ_η, which lives on input positions in $\eta(p)$ by C-2. This completes the proof that r_φ is a realization function.

We now show that the two substitutions commute: $\sigma_\theta\sigma_\eta = \sigma_\eta\sigma_\theta$. By C-2 for $\theta(p)$ and $\eta(p)$, the substitutions σ_θ and σ_η live on input positions in $\theta(p)$ and $\eta(p)$, respectively. These input positions do not overlap since $\theta(p) \to \eta(p)$ is a subformula of $X(p)$, which is properly annotated by H-1. Hence, each variable is changed by at most one of these substitutions. Clearly, if neither substitution changes x_i, then neither composition changes it either, so that $x_i\sigma_\theta\sigma_\eta = x_i\sigma_\eta\sigma_\theta = x_i$. If one of the substitutions, call it σ, changes x_i, the other substitution, call it τ, changes neither x_i nor $x_i\sigma$, the latter because the only variable it contains is x_i by C-3 for one of $\theta(p)$ or $\eta(p)$. Hence, $x_i\sigma\tau = x_i\sigma = x_i\tau\sigma$. It follows that $\sigma_\varphi = \sigma_\theta\sigma_\eta = \sigma_\eta\sigma_\theta$.

C-2. By C-2 for $\theta(p)$ and for $\eta(p)$, we have $x_i\sigma_\varphi = x_i\sigma_\theta\sigma_\eta = x_i$ if x_i is not in an input position in $\theta(p) \to \eta(p)$.

C-3 for $\varphi(p)$ easily follows from C-3 for $\theta(p)$ and $\eta(p)$.

C-4. Suppose r_1 is non-self-referential on variables over $X(B)$. Then both r_θ and r_η are so, too, by C-4 for $\theta(p)$ and $\eta(p)$. Hence, a variable x_k does not occur in $r_1(n)$, in $r_\theta(n)$, or in $r_\eta(n)$ for any $\Box_n$ occurring in $Z(B)$ for some subformula $\Box_{2k}Z(B)$ of $X(B)$. Since, by C-3 for $\theta(p)$ and $\eta(p)$, neither σ_θ nor σ_η introduces new variables, the variable x_k does not occur in $r_\theta(n)\sigma_\eta$ or in $r_\eta(n)\sigma_\theta$ either. Thus, x_k does not occur in $r_\varphi(Z(B))$ and r_φ is non-self-referential on variables over $X(B)$.

C-1. The statement that needs to be demonstrated depends on the polarity of this occurrence of $\theta(p) \to \eta(p)$. We only show one of the two cases since the other is analogous. Let this occurrence be positive. The the corresponding occurrence of $\theta(p)$ is negative and the corresponding occurrence of $\eta(p)$ is positive. In order to show C-1, i.e.,

$$\mathsf{JPAL(K)} \vdash [r_1(\theta(A))\sigma_\varphi \to r_1(\eta(A))\sigma_\varphi] \to [r_\varphi(\theta(B)) \to r_\varphi(\eta(B))].$$

It is sufficient to show

$$\mathsf{JPAL(K)} \vdash r_\varphi(\theta(B)) \to r_1(\theta(A))\sigma_\varphi \tag{15}$$

$$\mathsf{JPAL(K)} \vdash r_1(\eta(A))\sigma_\varphi \to r_\varphi(\eta(B)). \tag{16}$$

By C-1 for this negative occurrence of $\theta(p)$, $\mathsf{JPAL(K)} \vdash r_\theta(\theta(B)) \to r_1(\theta(A))\sigma_\theta$. Hence, $\mathsf{JPAL(K)} \vdash r_\theta(\theta(B))\sigma_\eta \to r_1(\theta(A))\sigma_\theta\sigma_\eta$ by the Substitution Lemma, which can be rewritten as $\mathsf{JPAL(K)} \vdash r_\varphi(\theta(B)) \to r_1(\theta(A))\sigma_\varphi$. This yields (15). By C-1 for this positive occurrence of $\eta(P)$, $\mathsf{JPAL(K)} \vdash r_1(\eta(A))\sigma_\eta \to r_\eta(\eta(B))$. Hence, by the Substitution Property, $\mathsf{JPAL(K)} \vdash r_1(\eta(A))\sigma_\eta\sigma_\theta \to r_\eta(\eta(B))\sigma_\theta$, which can be rewritten as $\mathsf{JPAL(K)} \vdash r_1(\eta(A))\sigma_\varphi \to r_\varphi(\eta(B))$. This yields (16).

Modal case. If $\varphi(p) = \Box_i\theta(p)$ and $\langle r_\theta, \sigma_\theta\rangle$ has been constructed. Unlike in other cases, this subformula, due to $\Box_i$ in it, can only occur once in $X(p)$. Accordingly we can talk about the goodness and polarity of this subformula rather than about those of its occurrences. The parity of i determines whether $\varphi(p)$ is a positive or a negative subformula. We show that $\langle r_\varphi, \sigma_\varphi\rangle$ witnesses the goodness of the subformula of $\Box_i\theta(p)$ in $X(p)$ under the assumption that $\langle r_\theta, \sigma_\theta\rangle$ witnesses the goodness of the corresponding occurrence of $\theta(p)$. In the modal case, the construction of r_φ and σ_φ depends on the polarity of $\varphi(p)$ in $X(p)$.

Modal subcase 1. If $\Box_i\theta(p)$ is a positive subformula of $X(p)$ so that i is odd, set $\sigma_\varphi := \sigma_\theta$. By C-1 for this positive occurrence of $\theta(p)$, $\mathsf{JPAL(K)} \vdash$

$r_1(\theta(A))\sigma_\theta \to r_\theta(\theta(B))$. By Internalization, construct a ground term u_1 such that

$$\mathsf{JPAL(K)} \vdash u_1 : [r_1(\theta(A))\sigma_\theta \to r_\theta(\theta(B))] \tag{17}$$

and set

$$r_\varphi(n) := \begin{cases} [u_1 \cdot r_1(i)]\sigma_\theta & \text{if } n = i \\ r_\theta(n) & \text{otherwise.} \end{cases}$$

C-2 and C-3 for $\varphi(p)$ easily follow from C-2 and C-3 for $\theta(p)$ because $\sigma_\varphi = \sigma_\theta$. For C-2, it is sufficient to note that any $\Box_{2j}$ occurring in $\theta(p)$ also occurs in $\Box_i\theta(p)$.

C-4. Suppose r_1 is non-self-referential on variables over $X(B)$. Then r_θ is so, too, by C-4 for $\theta(p)$. Hence, a variable x_k does not occur in either $r_1(Z(B))$ or $r_\theta(Z(B))$ for any subformula $\Box_{2k}Z(B)$ of $X(B)$. The only change from $r_\theta(Z(B))$ to $r_\varphi(Z(B))$ happens in the realization of $\Box_i$ if $\Box_i$ occurs in $Z(B)$: $r_\theta(i)$ becomes $[u_1 \cdot r_1(i)]\sigma_\theta$. Given that, in this case, x_k does not occur in $r_1(i)$, u_1 contains no variables, and σ_θ does not introduce new variables by C-3 for $\theta(p)$, the variable x_k does not occur in $r_\varphi(Z(B))$. Hence, r_φ is non-self-referential on variables over $X(B)$.

C-1. $\mathsf{JPAL(K)} \vdash (r_1(i)\sigma_\theta) : [r_1(\theta(A))\sigma_\theta] \to (u_1 \cdot (r_1(i)\sigma_\theta)) : r_\theta(\theta(B))$ follows from (17) by the application axiom and MP. Since

$$(r_1(i)\sigma_\theta) : [r_1(\theta(A))\sigma_\theta] = [r_1(i) : r_1(\theta(A))]\sigma_\theta = r_1(\Box_i\theta(A))\sigma_\theta = r_1(\varphi(A))\sigma_\theta \ ,$$

we have $\mathsf{JPAL(K)} \vdash r_1(\varphi(A))\sigma_\theta \to (u_1 \cdot (r_1(i)\sigma_\theta)) : r_\theta(\theta(B))$. Since u_1 contains no variables, $\mathsf{JPAL(K)} \vdash r_1(\varphi(A))\sigma_\theta \to [u_1 \cdot r_1(i)]\sigma_\theta : r_\theta(\theta(B))$. Since $\Box_i\theta(B)$ is a subformula of $X(B)$, which is properly annotated by H-1, the index i does not occur in $\theta(B)$ so that $r_\varphi(\theta(B)) = r_\theta(\theta(B))$. Thus, given that $\sigma_\varphi = \sigma_\theta$ and $r_\varphi(i) = [u_1 \cdot r_1(i)]\sigma_\theta$, we obtain $\mathsf{JPAL(K)} \vdash r_1(\varphi(A))\sigma_\varphi \to r_\varphi(i) : r_\varphi(\theta(B))$. Hence, $\mathsf{JPAL(K)} \vdash r_1(\varphi(A))\sigma_\varphi \to r_\varphi(\Box_i\theta(B))$, or

$$\mathsf{JPAL(K)} \vdash r_1(\varphi(A))\sigma_\varphi \to r_\varphi(\varphi(B)),$$

which is C-1 for the positive subformula $\varphi(p)$.

Modal subcase 2. If $\Box_i\theta(p)$ is a negative subformula of $X(p)$ so that $i = 2j$ is even, set $r_\varphi := r_\theta$. By C-1 for this negative occurrence of $\theta(p)$, $\mathsf{JPAL(K)} \vdash r_\theta(\theta(B)) \to r_1(\theta(A))\sigma_\theta$. By Internalization, construct a ground term u_1 such that

$$\mathsf{JPAL(K)} \vdash u_1 : [r_\theta(\theta(B)) \to r_1(\theta(A))\sigma_\theta] \tag{18}$$

and set

$$x_n\sigma_\varphi := \begin{cases} u_1 \cdot x_j & \text{if } n = j \\ x_n\sigma_\theta & \text{otherwise.} \end{cases}$$

C-4 for $\varphi(p)$ trivially follows from C-4 for $\theta(p)$.

C-2 and C-3. Since σ_φ differs from σ_θ only on x_j, it is sufficient to note two things. First, the only variable in $x_j\sigma_\varphi = u_1 \cdot x_j$ is x_j so that C-3 for $\varphi(p)$ follows from C-3 for $\theta(p)$. Second, x_j is in input position in $\Box_{2j}\theta(p)$ and any input position in $\theta(p)$ is also an input position in $\Box_{2j}\theta(p)$ so that C-2 for $\varphi(p)$ follows from C-2 for $\theta(p)$.

C-1. $\mathsf{JPAL(K)} \vdash x_j : r_\theta(\theta(B)) \to u_1 \cdot x_j : r_1(\theta(A))\sigma_\theta$ follows from (18) by the application axiom and MP. Since $r_\varphi(\varphi(B)) = r_\theta(\Box_{2j}\theta(B)) = x_j : r_\theta(\theta(B))$ and $x_j\sigma_\varphi = u_1{\cdot}x_j$, $\mathsf{JPAL(K)} \vdash r_\varphi(\varphi(B)) \to (x_j\sigma_\varphi) : r_1(\theta(A))\sigma_\theta$. Since $\Box_{2j}\theta(A)$ is a subformula of $X(A)$ and r_1 is non-self-referential on variables over $X(A)$ by H-2, the variable x_j does not occur in $r_1(\theta(A))$. Consequently,

$$(x_j\sigma_\varphi) : r_1(\theta(A))\sigma_\theta = (x_j\sigma_\varphi) : r_1(\theta(A))\sigma_\varphi = \\ [x_j : r_1(\theta(A))]\sigma_\varphi = [r_1(\Box_{2j}\theta(A))]\sigma_\varphi = [r_1(\varphi(A))]\sigma_\varphi$$

Thus, we have obtained C-1 for the negative subformula $\varphi(p)$:

$$\mathsf{JPAL(K)} \vdash r_\varphi(\varphi(B)) \to r_1(\varphi(A))\sigma_\varphi \ .$$

Announcement case. If $\varphi(p) = [\theta]\eta(p)$ and $\langle r_\eta, \sigma_\eta \rangle$ has been constructed, set $\sigma_\varphi := \sigma_\eta$ and

$$r_\varphi(n) := \begin{cases} r_\eta(n) & \text{if } \Box_n \text{ occurs in } \eta(B) \\ r_1(n) & \text{otherwise.} \end{cases}$$

Recall that p does not occur within announcements like θ. We show that $\langle r_\varphi, \sigma_\varphi \rangle$ witnesses the goodness of an occurrence of $[\theta]\eta(p)$ in $X(p)$ under the assumption that $\langle r_\eta, \sigma_\eta \rangle$ witnesses the goodness of the corresponding occurrence of $\eta(P)$. It is easy to see that r_φ is a realization function.

C-2 and C-3 for $\varphi(p)$ easily follow from C-2 and C-3 for $\eta(p)$ because $\sigma_\varphi = \sigma_\eta$. For C-2, it is sufficient to note that any $\Box_{2j}$ occurring in $\eta(p)$ also occurs in $[\theta]\eta(p)$.

C-4. Suppose r_1 is non-self-referential on variables over $X(B)$. Then r_η is too by C-4 for $\eta(p)$. Hence, a variable x_k does not occur in either $r_1(n)$ or $r_\eta(n)$ for any $\Box_n$ occurring in $Z(B)$ for some subformula $\Box_{2k}Z(B)$ of $X(B)$. Thus, x_k does not occur in $r_\varphi(Z(B))$ and r_φ is non-self-referential on variables over $X(B)$.

C-1. The statement that needs to be demonstrated depends on the polarity of this occurrence of $[\theta]\eta(p)$. We only show one of the two cases since the other is analogous. Let this occurrence be positive. Then the corresponding occurrence of $\eta(p)$ is also positive. In order to show C-1, i.e., given that θ contains no modalities, to show $\mathsf{JPAL(K)} \vdash [\theta]r_1(\eta(A))\sigma_\varphi \to [\theta]r_\varphi(\eta(B))$, it is sufficient to apply the admissible announcement necessitation rule, the normality axiom, and MP to $\mathsf{JPAL(K)} \vdash r_1(\eta(A))\sigma_\varphi \to r_\varphi(\eta(B))$, which is the same as $\mathsf{JPAL(K)} \vdash r_1(\eta(A))\sigma_\eta \to r_\eta(\eta(B))$, i.e., as C-1 for this positive occurrence of $\eta(p)$. $\Box$

After proving this theorem, general enough to carry the induction through, we formulate a weaker statement we are going to use to prove realization:

Corollary 41 (Replacement for $\mathsf{JPAL(K)}$)**.** *Assume the following:*

1. *A proposition p has exactly one occurrence in a properly annotated formula $X(p)$ that is outside of announcements. $X(A)$ and $X(B)$ are properly annotated formulas with no modalities within announcements.*
2. *r_1 is a realization function, non-self-referential on variables over $X(A)$ and over $X(B)$.*
3. *If p occurs positively in $X(p)$, $\mathsf{JPAL(K)} \vdash r_1(A) \to r_1(B)$.*
   If p occurs negatively in $X(p)$, $\mathsf{JPAL(K)} \vdash r_1(B) \to r_1(A)$.

Then there exists a realization function r and a substitution σ such that

$$\mathsf{JPAL(K)} \vdash r_1(X(A))\sigma \to r(X(B))$$

and r is non-self-referential on variables over $X(B)$.

It remains to extend the notions of one-step reduction and of reduction to annotated modal formulas (with announcements). To achieve this it is sufficient to replace the 4th row in the table in (3) by

Redex	Its reduct
$[A]\Box_i B$	$\Box_i(A \to [A]B)$

(19)

The functions red_1 and red for annotated formulas are defined the same way as in Definition 5 but based on the new set of reductions. Accordingly, $\mathsf{red}_1(A)$ is an annotated formula and $\mathsf{red}(A)$ is an annotated formula without announcements whenever A is an annotated formula.

Since the only difference in how red_1 works on $\mathsf{Fml}_{\Box,[\cdot]}$-formulas and on annotated formulas is such that erasing annotations in a pair of redex/reduct of the annotated red_1 yields a pair of redex/reduct of the unannotated red_1, the following lemma is not very surprising:

Lemma 42. *Let D be a properly annotated variant of $D' \in \mathsf{Fml}_{\Box,[\cdot]}$ and let neither one contain modalities within announcements.*[3] *Then $\mathsf{red}_1(D)$ and $\mathsf{red}(D)$ are properly annotated variants of $\mathsf{red}_1(D')$ and $\mathsf{red}(D')$ respectively, neither $\mathsf{red}_1(D)$ nor $\mathsf{red}_1(D')$ contains modalities within announcements, and neither $\mathsf{red}(D)$ nor $\mathsf{red}(D')$ contains announcements.*

Proof. That $\mathsf{red}_1(D)$ is an annotated version of $\mathsf{red}_1(D')$ and, hence, $\mathsf{red}(D)$ is an annotated version of $\mathsf{red}(D')$ is clear from the definition. Thus, it remains to prove that red_1 preserves the properness of annotations and the property of not having modalities within announcements. We will only show this for the pair of redex/reduct from (19). The other four cases are even simpler. Let $D = X([A]\Box_i B)$ for some $X(p)$ and red_1 maps it to $X(\Box_i(A \to [A]B))$. By assumption, A contains no modalities and p is outside of announcements in $X(p)$. Hence, after the replacement $\Box_i$ and the modalities in B remain outside of announcements. Further, all annotations in $\Box_i(A \to [A]B)$, i.e., i and all annotations in B, do not occur in $X(p)$ because D is properly annotated. The duplication of A does not violate the properness of annotation because A is modality-free. $\Box$

We now have all the ingredients sufficient to establish our realization theorem. The following diagram shows how we obtain it. We start with a formula $D' \in \mathsf{Fml}_{\Box,[\cdot]}$. Taking its arbitrary properly annotated version D, using annotated reduction from Lemma 42, K realization from Theorem 39, and replacement from Corollary 41, we construct a formula $r(D) \in \mathsf{Fml}_{\mathsf{J}}$ that realizes D. It is easy to see that $(r(D))^\circ = D'$.

$$\begin{tikzcd}
D \arrow[d, "\text{Reduction}"'] & r(D) \arrow[l, dashed, "\text{Forgetful projection}"'] \\
\mathsf{red}(D) \arrow[r, "\mathsf{K}\text{ realization}"'] & r_{\mathsf{K}}(\mathsf{red}(D)) \arrow[u, "\text{Replacement to `invert' }\mathsf{red}"']
\end{tikzcd}$$

Theorem 43 (Realization for $\mathsf{PAL(K)}$)**.** *If D' is a theorem of $\mathsf{PAL(K)}$ that does not contain modalities within announcements, then for any properly annotated version D of D', there is a realization function r such that $r(D)$ is provable in $\mathsf{JPAL(K)}$ and $(r(D))^\circ = D'$.*

Proof. Let D be a properly annotated version of D'. Clearly, D contains no modalities within announcements. By Lemma 42, there exists a sequence of properly annotated formulas $D_0, D_1, \dots, D_N$ that do not contain modalities within announcements and a sequence of $\mathsf{Fml}_{\Box,[\cdot]}$-formulas

[3] If one does not contain them, then the other does not either.

$D'_0, D'_1, \ldots, D'_N$ that do not contain modalities within announcements such that $D_0 = D$,$D'_0 = D'$, $D_{i+1} = \mathsf{red}_1(D_i)$ and $D'_{i+1} = \mathsf{red}_1(D'_i)$ for each $i = 0, \ldots, N-1$, $D_N = \mathsf{red}(D)$, $D'_N = \mathsf{red}(D')$, neither D_N nor D'_N contains announcements, and D_i is a properly annotated version of D'_i for each $i = 0, \ldots, N$. By Lemma 6, $\mathsf{PAL(K)} \vdash D' \leftrightarrow D'_N$. Hence, D'_N is also a theorem of $\mathsf{PAL(K)}$. Given that D'_N contains no announcements, it is also a theorem of K due to the conservativity of $\mathsf{PAL(K)}$ over K. By Theorem 39, there exists a realization function r_N that is non-self-referential on variables over the properly annotated version D_N of D'_N such that $r_N(D_N)$ is provable in J and, hence, in its extension $\mathsf{JPAL(K)}$. We now construct realization functions r_{N-i}, non-self-referential on variables over D_{N-i}, such that $r_{N-i}(D_{N-i})$ is a theorem of $\mathsf{JPAL(K)}$ for $i = 0, \ldots, N$ by induction on i. The base case is already established. Let $r_{N-(i-1)} = r_{N-i+1}$ be already constructed. Since $D_{N-i+1} = \mathsf{red}_1(D_{N-i})$, it follows that $D_{N-i} = X(\mathit{Redex})$ for some $X(p)$ with exactly one occurrence of a fresh proposition p whereas $D_{N-i+1} = X(\mathit{Reduct})$, where *Redex* is the outermost leftmost redex in D_{N-i} and *Reduct* is the reduct of *Redex*. We want to apply Corollary 41 to $X(p)$, *Reduct*, *Redex*, and r_{N-i+1}. Assumption 1 is satisfied because D_{N-i} and D_{N-i+1} are properly annotated and by definition of red_1 (recall also that an outermost redex never occurs within announcements). That r_{N-i+1} is non-self-referential on variables over $X(\mathit{Reduct}) = D_{N-i+1}$ follows from the induction hypothesis. Looking at the five types of redexes and their reducts, it is easy to check that r_{N-i+1} is also non-self-referential on variables over $X(\mathit{Redex})$ because the substitution of *Redex* for *Reduct* at p in $X(p)$ never introduces new modalities and never moves modalities into the scope of other modalities (recall that announcements contain no modalities). Hence, assumption 2 is satisfied. It remains to note that $r(\mathit{Redex}) \leftrightarrow r(\mathit{Reduct})$ is one of the axioms of $\mathsf{JPAL(K)}$[4] for any realization function r, including r_{N-i+1}, so that assumption 3 is also satisfied, independent of the polarity of p in $X(p)$. By Corollary 41, there exists a realization function r_{N-i}, non-self-referential on variables over $X(\mathit{Redex}) = D_{N-i}$, and a substitution σ_i such that $\mathsf{JPAL(K)} \vdash r_{N-i+1}(X(\mathit{Reduct}))\sigma_i \to r_{N-i}(X(\mathit{Redex}))$. In other words, $\mathsf{JPAL(K)} \vdash r_{N-i+1}(D_{N-i+1})\sigma_i \to r_{N-i}(D_{N-i})$. It remains to use the induction hypothesis, the Substitution Property, and MP to see that $r_{N-i}(D_{N-i})$ is a theorem of $\mathsf{JPAL(K)}$. In particular, $r_{N-N}(D_{N-N}) = r_0(D)$ is provable in $\mathsf{JPAL(K)}$. Set $r := r_0$. Clearly, $(r(D))^\circ = D'$. □

Remark 44. It is not clear how to generalize our proof to theorems of $\mathsf{PAL(K)}$ with modalities allowed within announcements. The problem is that a reduct $[\Box A]C \to [\Box A]D$ has two copies of $\Box A$, which need to be combined into only

[4]This is not the case in $\mathsf{OPAL(K)}$ for redex/reduct pairs from (19).

one copy in the corresponding redex $[\Box A](C \to D)$. In general, the outer $\Box$ in $\Box A$'s in the reduct will be realized by different terms, and we currently lack methods of merging terms within announcements.

Remark 45. Unfortunately, adapting this proof to $\mathsf{OPAL(K)}$ presents certain challenges. The problem is that in order to 'invert' the reduction from $\mathsf{PAL(K)}$ to K, we need to apply replacement also in negative positions. This is only possible because in the update axiom (update) of $\mathsf{JPAL(K)}$, we have the same evidence term on both sides of the equivalence. If, like in $\mathsf{OPAL(K)}$, we work with update operations $\Uparrow$ and $\Downarrow$ on terms, then we end up with different terms in the update axioms, which prevents the use of Fitting's replacement at negative positions.

7 Conclusion

We present $\mathsf{JPAL(K)}$ and $\mathsf{OPAL(K)}$, two alternative justification logic counterparts of Gerbrandy–Groeneveld's modal public announcement logic $\mathsf{PAL(K)}$. One of $\mathsf{PAL(K)}$'s update principles is

$$\Box(A \to [A]B) \to [A]\Box B \ ,$$

which we render in $\mathsf{JPAL(K)}$ as

$$s : (A \to [A]B) \to [A]s : B$$

and in $\mathsf{OPAL(K)}$ as

$$s : (A \to [A]B) \to [A] \Uparrow s : B \ .$$

For the semantics, we use a combination of the traditional semantics for public announcement logic (where an agent rejects as impossible the worlds that are inconsistent with the announcement made) and evidence functions from epistemic models for justification logic (that specify for each world which formulas an evidence term can justify). We then show soundness and completeness (by a canonical model construction) for $\mathsf{JPAL(K)}$ and $\mathsf{OPAL(K)}$.

The main result of the paper is a realization theorem stating that $\mathsf{JPAL(K)}$ realizes all the theorems of $\mathsf{PAL(K)}$ that do not contain modalities within announcements. To obtain this result we have to extend Fitting's replacement theorem such that, first, it works in the context of public announcements and, second, it allows replacement also in negative positions.

Finally, it should be noted that our novel realization method does not rely on a cut-free deductive system for $\mathsf{PAL(K)}$. Its constructiveness, however, depends on constructive realization for the modal logic K, to which we reduce $\mathsf{PAL(K)}$.

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